Journal of Engineering Mathematics Vol. II, Nr. 2, April 1968.
Wolters-Noordhoff Publishing - Groningen
Printed in the Netherlands.

TRANSIENT THERMAL BEHAVIOUR OF GROUND ELECTRODES

Part I: SPHERICAL ELECTRODES

by

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1. *Introduction*

HVDC transmission projects may be designed with earth return - either as a permanent mode of operation, or an emergency provision. Operation with earth return requires a set of ground electrodes which may have to carry heavy DC currents. If one or both of the HVDC terminals are located near the sea, electrode arrangements may be exposed directly to sea water. In this case no electrode temperature problem really exists. On the other hand, a so-called land electrode which should be capable of carrying a high current to ground over an extended period of time, must be designed so that its temperature rise does not reach a dangerous level.

Temperature rise near an electrode embedded in soil will affect the physical parameters of the soil. Of particular importance in this connection is soil moisture movement caused by the electric and the thermal fields. One may assume, however, that unless the soil dries out, no radical change will occur in the soil parameters as a result of temperature rise. Should excessive heating take place somewhere in the system, the result would be an increase in soil resistivity. This situation is potentially dangerous. The maximum temperature in the electrode system should therefore not exceed a certain limit, below 100°C.

In the steady state the temperature field is governed by Laplace's Equation just as the electric potential field. Since similar boundary conditions may be assumed for the two fields, it is possible to express the temperature rise (steady state) of the electrode in terms of its electric potential, and electrical and thermal parameters of the surrounding medium.[1,2] The relationship so expressed is valid for any electrode configuration.

The question naturally arises as to how long it takes the electrode, or a point outside of it, to reach final temperature, and what sort of function governs the temperature rise. This paper will attempt to present an answer, qualified by the assumptions described below.

In literature on this subject Thermal Time Constant for the electrode is sometimes used as a practical parameter to describe its thermal behaviour [1,3,4,5] . It should perhaps be pointed out that the meaning of time constant is not unique except for first order systems. Thermal Time Constant may be defined as final temperature rise divided by initial rate of temperature rise. It may also be defined as the time required to reach $(1 - e^{-1})$ times final temperature rise. These two definitions become identical when the temperature-time function is purely exponential. In the case of ground electrodes, the temperature-time dependence follows a different pattern and the two definitions of Time Constant therefore yield different values.

It seems that if one were to use a characteristic parameter of this kind, the second definition would be more meaningful. It has been adopted in this paper, and the corresponding parameter is called Apparent Time Constant, τ_A.

2. *Assumptions*

The investigation of the thermal behaviour of the electrodes has been based on several simplifying assumptions. First of all, an electrode arrangement which allows radial symmetry is assumed. Two general cases involving spherical electrodes will be discussed in this paper: 1) concentric spheres, and 2) a sphere in an infinite medium. It is recognized that these arrangements only vaguely resemble real ground electrodes. On the other hand, the results obtained for them will indicate certain features that are to be expected for other arrangements as well.

In the case of concentric spheres, the boundary conditions have been assumed as follows:

1. A step rise in electric potential from zero to U_0 occurs at the inner sphere or shell at time $t = 0$.
2. The outer sphere is held at reference or zero potential all the time.
3. The inner sphere has negligible heat capacity compared to the near environment, hence there is no heat flow across the surface of the inner sphere.
4. The temperature of the outer sphere remains constant, independent of time.
5. Initially, the medium is at uniform temperature equal to ambient.

It is further assumed that the medium between the two spheres is homogeneous and isotropic, and that its electrical and thermal properties are temperature independent. Soil does not fully satisfy these conditions. Significant deviation from them must be taken into account when evaluating the results of the analysis.

Boundary Condition #1 is equivalent to specifying constant electrode current (for a given electrode) provided the electrical resistivity of the soil does not change with temperature.

The single sphere problem results by letting the radius of the outer sphere become infinitely large. Subject to this modification, boundary conditions remain the same in both cases.

3. *Analysis*

A. Concentric Spheres

Consider the arrangement of concentric spheres as illustrated in Fig. 1. The potential at radius b remains zero while that of the inner sphere at radius a rises from zero to U_0 at time $t = 0$.

The potential distribution between the spheres satisfies Laplace's Equation. Taking into account the boundary and symmetry conditions, the potential at radius r becomes

$$U(r) = U_0 \left[\frac{ab}{b-a} \frac{1}{r} - \frac{a}{b-a} \right] \tag{1}$$

Power loss due to current flow through the medium results in heat generation. The Diffusion Equation which governs heat transfer and describes the thermal field must therefore include a heat generation term. It can be shown that the partial differential equation for the thermal field is

$$\nabla^2 T + \rho_t \, (\mathbf{E} . \mathbf{i}) = \frac{1}{k} \frac{\partial T}{\partial t} \tag{2}$$

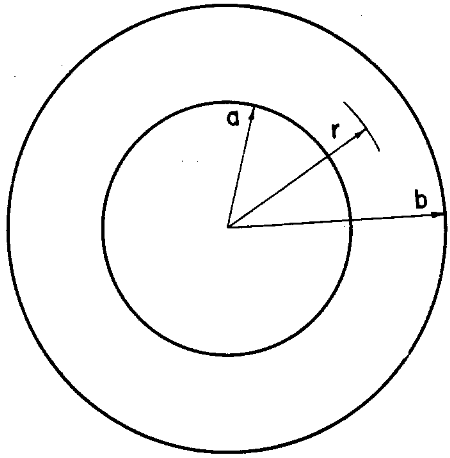

Fig. 1. Concentric spheres.

where $T = T(r,t)$, temperature, ρ_t = thermal resistivity, $\mathbf{E}$ and $\mathbf{i}$ electric field intensity and current density vectors respectively. Since

$$\mathbf{E} = -\nabla U = U_o \frac{ab}{b-a} \frac{\mathbf{r}}{r^2}$$

and $\mathbf{i} = \frac{1}{\rho_e} \mathbf{E}$, ρ_e = electric resistivity, it follows that the generation term may be expressed as

$$\rho_t(\mathbf{E}.\mathbf{i}) = \frac{\rho_t}{\rho_e} |\mathbf{E}|^2 = \frac{\rho_t}{\rho_e} \left(\frac{U_o\, ab}{b-a}\right)^2 \frac{1}{r^4} \tag{3}$$

For convenience, a parameter G_s has been defined as

$$G_s = \left[\frac{U_o\, ab}{b-a}\right]^2 \rho_t/\rho_e$$

and the heat generation term in the problem involving spheres is given in terms of G_s and r, i.e.

$$\rho_t(\mathbf{E}.\mathbf{i}) = G_s/r^4$$

Under conditions of radial symmetry, the partial differential equation for T may therefore be written

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{G_s}{r^4} = \frac{1}{k} \frac{\partial T}{\partial t} \tag{4}$$

Solution of equation (4) is obtained using a method which involves separation of variables and functions. One can assume the following form of the solution:

$$T(r,t) = R(r)\, S(t) + f(r) \tag{5}$$

where $R(r)$ and $f(r)$ are radius dependent and $S(t)$ time dependent only. The time dependent part of the solution must vanish as t approaches infinity. $f(r)$ therefore represents the steady-state temperature distribution.

Combining equations (4) and (5) yields

$$\frac{S}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{r^2}+\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right)+\frac{G_s}{r^4}=\frac{R}{k}\frac{\partial S}{\partial t}\ . \tag{6}$$

The general requirements are satisfied if in equation (6) the sum of the f- and G_s-terms is set to zero, i.e.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right)+\frac{G_s}{r^4}=0 \tag{7}$$

and hence

$$\frac{S}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial t}\right)=\frac{R}{k}\frac{\partial S}{\partial t} \tag{8}$$

The formal solution of (7) is found as

$$f(r)=-\frac{G_s}{2r^2}-\frac{G}{r}+C_2 \tag{9}$$

Since (9) represents the final or steady-state temperature distribution, it is subject to boundary conditions 3 and 4, viz.,

$$T\Big|_{r=b}=T_{amb}\ \text{and}\ \frac{\partial T}{\partial r}\Big|_{r=a}=0.$$

Applying these, the constants of integration are found and the result is

$$f(r)=T_{amb}+G_s\left[\frac{1}{ar}-\frac{1}{2r^2}-\frac{1}{ab}+\frac{1}{2b^2}\right] \tag{10}$$

Equation (8) can be re-arranged and written

$$\frac{1}{Rr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)=\frac{1}{kS}\frac{\partial S}{\partial t} \tag{11}$$

Both sides of (11) must equal the same constant, $\pm\lambda^2$, λ = real. The solution for the time dependent portion is readily obtained:

$$\frac{1}{kS}\frac{\partial S}{\partial t}=\pm\lambda^2$$

$$S=C_3\ e^{-\lambda^2kt} \tag{12}$$

The physical nature of the problem requires that the minus sign be chosen in the exponent of equation (12).

Expanding the left-hand side of (11), one obtains the differential equation for $R=R(r)$;

$$\frac{\partial^2R}{\partial r^2}+\frac{2}{r}\frac{\partial R}{\partial r}+\lambda^2R=0. \tag{13}$$

The solution of (13) has the general form

$$R = \frac{1}{r}\,(A\ \sin\lambda r + B\ \cos\lambda r) \tag{14}$$

Now, substituting for R, S and f in equation (5), using (10), (12) and (14), and combining the constants of integration, the formal solution for the temperature becomes

$$T(r,t) = \sum_{n=0}^{\infty} e^{-\lambda_n^2 kt}\ \frac{1}{r}\,(A_n \sin\lambda_n r + B_n \cos\lambda_n r) + T_{amb} + G_s\left(\frac{1}{ar} - \frac{1}{2r^2} - \frac{1}{ab} + \frac{1}{2b^2}\right) \tag{15}$$

As shown in Appendix 1 the eigenvalues are defined by the following equation:

$$\lambda_n a\ \cos\ \lambda_n(b-a) + \sin\ \lambda_n(b-a) = 0 \tag{16}$$

and the general solution may be expressed as

$$T(r,t) = \sum_{n=0}^{\infty} e^{-\lambda_n kt}\ C_n\ \frac{\sin\ \lambda_n(b-r)}{r} + f(r) \tag{17}$$

where f(r) is given by equation (10).
The coefficients C_n in equation (17) are defined by the relation (see Appendix 1)

$$C_n = \frac{2G_s}{b-a\ \sin^2\lambda_n(b-a)} \int_a^b \sin\ \lambda_n(b-r)\left[\frac{1}{2r} - \frac{1}{a} + \frac{r}{ab} - \frac{r}{2b^2}\right]dr$$

One of the integrals involved in determining C_n is transcendental and must be evaluated numerically. Also, an iterative method is required to determine the eigenvalues from equation (16). In order to find particular values of the function T(r, t) the use of a digital computer appears necessary. A program was written and used to obtain the numerical results for T(r, t) presented below.

B. Sphere in an Infinite Medium.

The technique of analysis employed in the study of concentric spheres requires a finite outer radius, b. As the ratio b/a increases, numerical evaluation of the solution (equation (17)) becomes increasingly difficult. Although the results indicate that Apparent Time Constant for the inner electrode converges to a definite value as b/a increases without limit, this (final) value of τ_A could not be predicted accurately even with maximum ratio b/a equal to 30. The single sphere problem has therefore been studied separately.

With basic assumptions as in the previous case, and taking $b = \infty$; the constant G_s becomes

$$G_s = (U_0\, a)^2\ \rho_t/\rho_e$$

The partial differential equation governing the thermal field remains unchanged, (equation (4)), with solution of the form given by equation (5). The f function which describes the final temperature distribution now becomes

$$f = T_{amb} + G_s \left(\frac{1}{ar} - \frac{1}{2r^2}\right) \tag{18}$$

Going back to the basic differential equation, it is convenient to introduce the functions $V = V(r, t)$ and $g = g(r)$, defined as follows:

$$T(r, t) = \frac{V(r, t)}{r} + \frac{g(r)}{r} \tag{19}$$

It will be noted that

$$g(r) = r\ f(r) \tag{20}$$

and that $V(r, t)$ must satisfy the Diffusion Equation, i.e.

$$\frac{\partial^2 V}{\partial r^2} = \frac{1}{k}\frac{\partial V}{\partial t} \tag{21}$$

Boundary Condition #3 applied to $V(r, t)$ requires that

$$\frac{\partial}{\partial r}\left(\frac{V}{r}\right)\bigg|_{r=a} = -\frac{V}{a^2} + \frac{1}{a}\frac{\partial V}{\partial r}\bigg|_{r=a} = 0 \tag{22}$$

If a function $S = S(r, t)$ is defined as

$$S = -V + a\frac{\partial V}{\partial r} \tag{23}$$

then Boundary Condition #3 implies that $S(a, t) = 0$, i.e. a homogeneous equation. The S-function also satisfies the Diffusion Equation. As shown in Appendix 2, the solution for S, taking into account appropriate boundary conditions, is

$$S(r, t) = \frac{G_s}{2\sqrt{\pi k t}} \int_0^{\infty} \left(\frac{1}{a} - \frac{1}{2(x^1+a)} - \frac{a}{2(x^1+a)^2}\right)\left(e^{\frac{-(r-a-x^1)^2}{4kt}} - e^{\frac{-(r-a+x^1)^2}{4kt}}\right) dx^1 \tag{24}$$

The next step is to find $V(r, t)$. According to equation (23),

$$\frac{S}{a} = -\frac{V}{a} + \frac{\partial V}{\partial r} \tag{25}$$

The homogeneous solution for V,

$$V = C\ e^{r/a},$$

is not applicable since V must remain bounded as r approaches infinity. Therefore $C = 0$. To obtain the particular solution, multiply equation (25) by $e^{-r/a}$ to give

$$\frac{S}{a}e^{-r/a} = -\frac{V}{a}e^{-r/a} + e^{-r/a}\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}(Ve^{-r/a}) \tag{26}$$

Integrating both sides between arbitrary limits c and d yields:

$$Ve^{-r/a}\bigg|_c^d = \frac{1}{a}\int_c^d Se^{-r/a}\ dr \tag{27}$$

By choosing the limits of integration in (27),

$$c = \infty$$
$$d = r$$

it can be shown that the solution obtained satisfies the (remaining) boundary condition, viz. temperature at infinite radius remains constant and equal to ambient - independent of time. The solution for V is thus

$$V = \frac{e^{r/a}}{a} \int_{\infty}^{r} S e^{-r/a} \, dr \tag{28}$$

Combining (28) with (19), (20) and (18), the solution for the temperature field can be expressed as follows:

$$T(r,t) = T_{amb} + G_s\left(\frac{1}{ar} - \frac{1}{2r^2}\right) - \frac{e^{r/a}}{ra} \int_{r}^{\infty} S(r,t) e^{-r/a} \, dr \tag{29}$$

- where $S(r,t)$ is given by equation (24)

The solution obtained involves a double integration. Because of the types of functions encountered, integration must be done numerically. A computer program was written and used to obtain the results presented below for the case of the single sphere.

Results

A. Concentric Spheres

The general solution given by equation (17) has been evaluated using physical parameters typical for soil, and an inner sphere radius a = 0.5 m. Various values of the ratio b/a were considered. Physical parameters used are:

Electric Resistivity,	ρ_e	= 100 ohm-m
Thermal " ,	ρ_t	= 1.0 m°C/Watt
Heat Capacity,	c	= 840 Watt-sec/(kg.°C)
Mass Density	γ	= 2500 kg/m^3

Hence Thermal Diffusivity $k = (\rho_t \gamma c)^{-1} = 0.475\,(10)^{-6}$ m/sec. For the purpose of the numerical evaluation, U_o is chosen as 122 volt and T_{amb} = 25°C. The final temperature of the electrode then becomes 100°C.

If the exponent of the time dependent part of equation (17) is written

$$\lambda_n^2 kt = \frac{t}{\tau_n}$$

i.e. $$\tau_n = \frac{1}{k\lambda_n^2} \quad (n^{th} \text{ time coefficient})$$

it appears convenient to normalize the time dependence by choosing τ_1 (fundamental time coefficient) as reference time unit.

The eigenvalue equation is given as

$$\lambda_n \, b \cos \lambda_n(b-a) + \sin \lambda_n(b-a) = 0$$

Let

$$\lambda_n(b-a) = \alpha_n, \quad \lambda_n = \alpha_n/(b-a)$$

where α_n is the n^{th} "dimensionless eigenvalue". Substituting for λ_n in the eigenvalue equation, one obtains

$$\alpha_n \frac{b}{a}\left(\frac{b}{a} - 1\right)^{-1} \cos \alpha_n + \sin \alpha_n = 0$$

By similar substitution the n^{th} time coefficient τ_n may be expressed in terms of α_n as

$$\tau_n = \frac{a^2}{k\alpha_n^2}\left(\frac{b}{a} - 1\right)^2$$

Thus it is seen that the dimensionless eigenvalues are functions of the *ratio* b/a only, and that for any given ratio b/a, the time coefficients are proportional to the square of the inner radius.

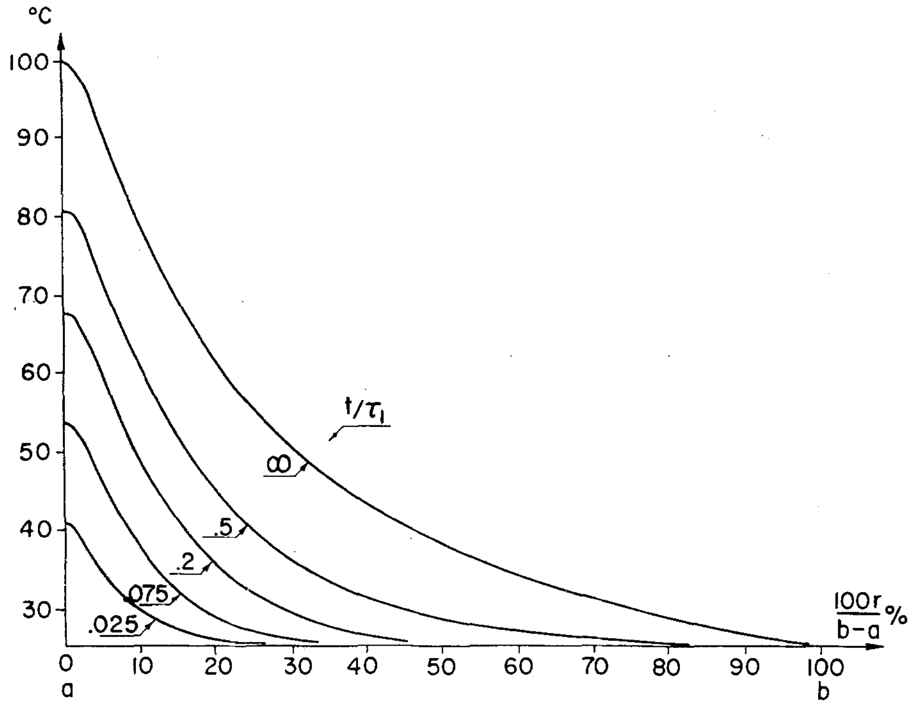

Fig. 2. Temperature distribution between concentric spheres with scaled time as parameter. b/a = 10.

Fig. 2 shows the temperature-radius dependence with scaled or relative time as parameter and for a ratio b/a = 10. In Fig. 3 the temperature at the inner electrode is plotted against relative time for a number of ratios of b/a. By determining the fundamental time coefficient, τ_1, the Apparent Time Constant may be found using Fig. 3 simply by multiplying the scaled time at which the temperature rise has reached $100(1 - e^{-1})\%$ of final value by τ_1. Apparent Time Constant τ_A as obtained in this manner is shown in Fig. 4. τ_A increases towards a definite value as the ratio

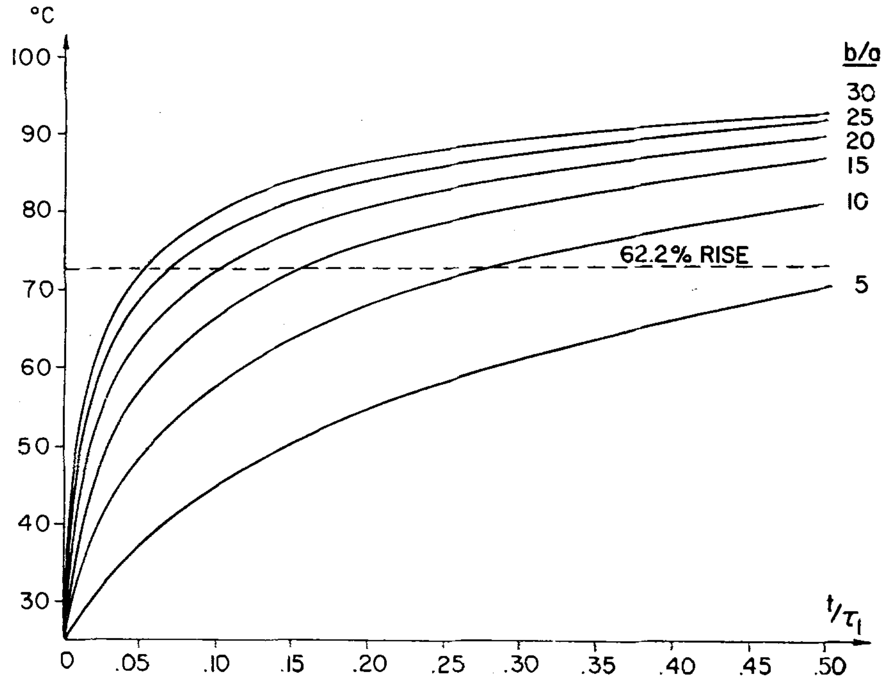

Fig. 3. Inner sphere temperature as a function of scaled time with ratio b/a as parameter. a = 0.5 m.

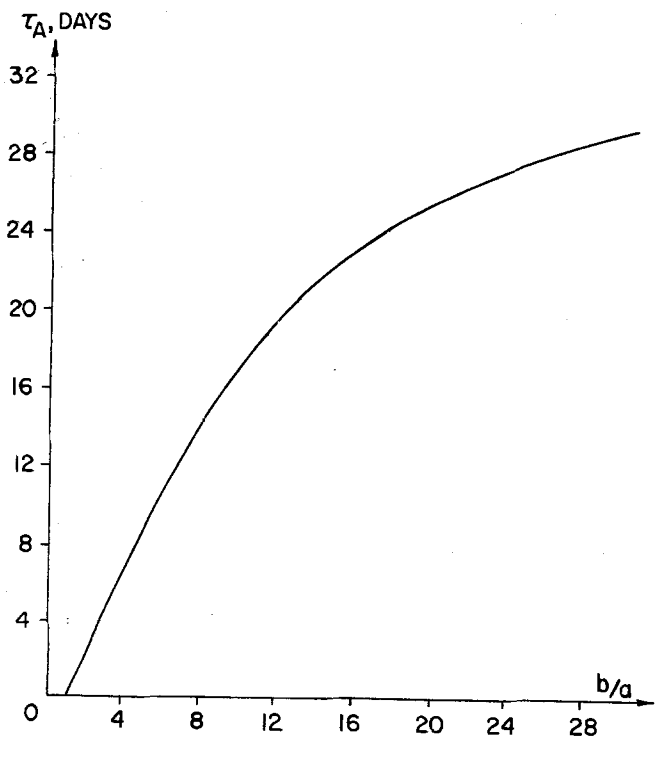

Fig. 4. Apparent Time Constant for the inner sphere as a function of b/a. a = 0.5 m.

b/a becomes very large. It is seen, however, that the limiting value of τ_A as b approaches infinity can not be predicted accurately unless the curve in Fig. 4 is extended to b/a = 40 or 50. Difficulties arise on the numerical side before this high ratio is reached.

B. Single Sphere

The general solution obtained for the single sphere does not yield the surface temperature directly because of singularity at r = a. Numerical results have been evaluated at points close to the surface, and the values for the surface obtained by extrapolation.

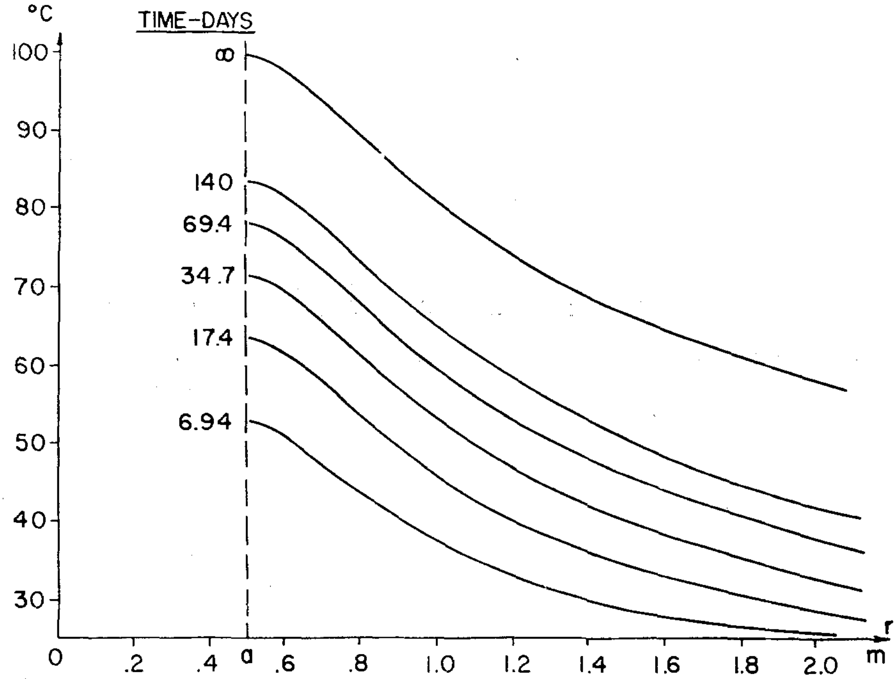

Fig. 5. Temperature distribution outside a spherical electrode with time as parameter. Electrode radius a = 0.5 m.

With physical parameter values as in the previous case, the temperature-radius relationship for various values of time is shown in Fig. 5. The sphere radius is 0.5 m. Fig. 6 illustrates the temperature-time function for the same spherical electrode and for a spherical electrode of radius 1.0 m. The Apparent Time Constant is approximately 38 days for the first electrode, and 155 days for the second. It has increased by a factor of about four as a result of doubling the sphere radius. Some indication to this effect was found in the solution for concentric spheres. Also it may be noted that τ_A = 38 days for the sphere of radius 0.5 m seems to agree well with projection based on Fig. 4 (as b/a $\rightarrow \infty$).

The initial rate of temperature rise is relatively high at the electrode surface. If the electrode temperature increased at this rate, final value would be reached in a matter of about 3 days in the case of the smaller sphere, and about 12.1 days for the larger. A "Time Constant" based on the initial rate of temperature rise therefore tends to be highly pessimestic if not misleading, assuming temperature independent parameters.

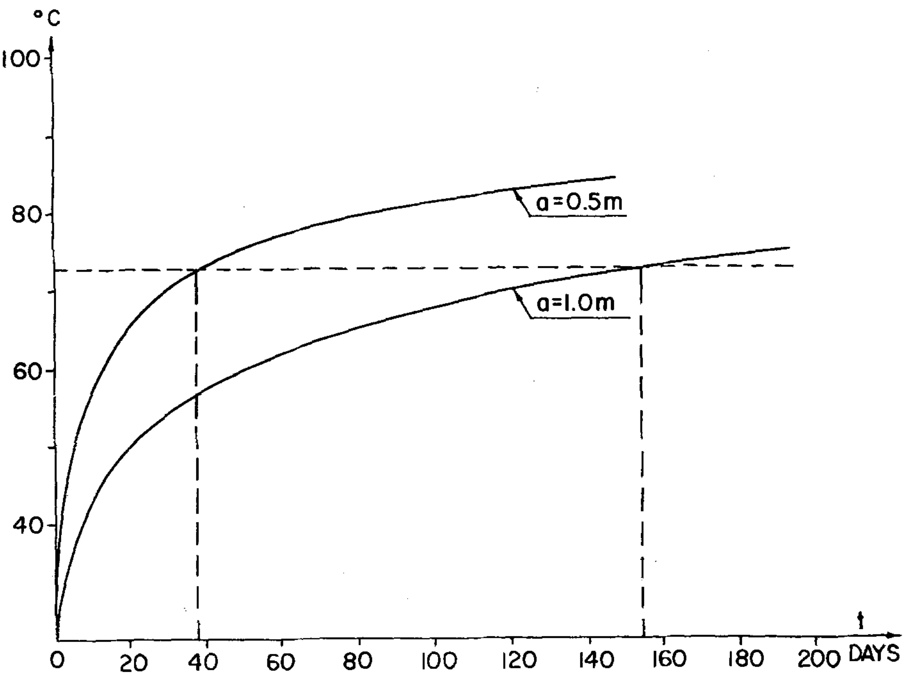

Fig. 6. Temperature of single sphere electrode as a function of time.

Conclusions

The transient thermal behaviour of spherical electrodes carrying direct current has been analysed by classical methods. Formal solutions have been obtained for two cases; one involving concentric spheres, the other a single sphere in an infinite medium. Numerical integration must be employed to obtain particular solutions and computer programs were developed for this purpose. Numerical results based on average physical conditions in moist soil show that the Apparent Time Constant is significantly higher than the value obtained using the initial rate of temperature rise.

Acknowledgements

The authors wish to acknowledge with thanks the financial assistance received from the National Research Council of Canada in support of this research project.

Appendix 1 Application of Boundary Conditions on T(r, t) for the Concentric Sphere Arrangement

Equation (14) represents the solution of the spatial equation (13):

$$R(r) = \frac{1}{r}\,(A \sin \lambda r + B \cos \lambda r)$$

The boundary conditions are applied to determine the eigenvalues λ_n and the corresponding ratios A_n/B_n.

At the inner boundary $\partial T/\partial r \,\Big|_{r=a} = 0$ is satisfied if $dR/dr \,\Big|_{r=a} = 0$, or

$$A(\lambda a \cos \lambda a - \sin \lambda a) - B(\lambda a \sin \lambda a + \cos \lambda a) = 0 \tag{30}$$

At the outer boundary $T(b,t) = T_{amb}$ is satisfied if $R(b) = 0$, or

$$A \sin \lambda b + B \cos \lambda b = 0 \tag{31}$$

From equations (30) and (31), one finds

$$\frac{A}{B} = -\frac{\cos \lambda b}{\sin \lambda b} \tag{32}$$

and

$$\lambda a \cos \lambda(b-a) + \sin \lambda(b-a) = 0 \tag{33}$$

Equation (33) defines the eigenvalues λ_n, $n = 0, 1, \ldots,$ where

$$n\pi < \lambda_n < (n+1)\pi$$

The coefficients B_n (or A_n) have yet to be determined. For this purpose the initial condition is applied. Combining equations (15) and (32) at $t = 0$ yields

$$T(r,0) = T_{amb} = \sum_{n=0}^{\infty} \frac{B_n}{r \sin \lambda_n b} (\sin \lambda_n b \cos \lambda_n r - \cos \lambda_n b \sin \lambda_n r) + T_{amb} + G_s\left(\frac{1}{ar} - \frac{1}{ab} - \frac{1}{2r^2} + \frac{1}{2b^2}\right)$$

i.e.

$$\sum_{n=0}^{\infty} C_n \frac{\sin \lambda_n(b-r)}{r} = -G_s\left(\frac{1}{ar} - \frac{1}{ab} - \frac{1}{2r^2} + \frac{1}{2b^2}\right) \tag{34}$$

where

$$C_n = B_n / \sin \lambda_n b$$

Multiplying equation (34) by r, one obtains

$$\sum_{n=0}^{\infty} C_n \sin \lambda_n(b-r) = G_s\left(\frac{1}{2r} - \frac{1}{a} + \frac{r}{ab} - \frac{r}{2b^2}\right) \tag{35}$$

It can be shown that the functions $\sin \lambda_n(b-r)$ are orthogonal over the interval (a,b) and that the normalization constant, β_n, is

$$\beta_n = \tfrac{1}{2}(b - a \sin^2 \lambda_n(b-a)) \tag{36}$$

The coefficients C_n are therefore found, using the orthogonality property, as

$$C_n = \frac{G_s}{\beta_n} \int_a^b \left(\sin \lambda_n(b-r)\right) \left[\frac{1}{2r} - \frac{1}{a} + \frac{r}{ab} - \frac{r}{2b^2}\right] dr \tag{37}$$

It will be noted that one of the integrals in (37), viz. $\int_a^b \frac{\sin \lambda_n(b-r)}{2r} dr$, is transcendental.

Appendix 2 Solution of the Function S(r, t)

S = S(r, t) is defined by equation (23) as

$$S = -V + a\frac{\partial V}{\partial r}$$

As has been shown above, the third Boundary Condition implies that S(a, t) = 0. Boundary Condition 5 states that T(r, 0) = T_{amb}, hence according to equation (19)

$$\left.\frac{V}{r}\right|_{t=0} + \left.\frac{g}{r}\right|_{t=0} = T_{amb} \tag{38}$$

Using equations (18) and (20), Boundary Condition 5 (initial condition) requires that

$$V(r, o) = G_s\left(\frac{1}{2r} - \frac{1}{a}\right) \tag{39}$$

With respect to S the initial condition is thus obtained on the basis of equations (23) and (39);

$$S(r, o) = G_s\left(\frac{1}{a} - \frac{1}{2r} - \frac{a}{2r^2}\right) \tag{40}$$

Let x = r - a, or r = x + a ; x $\geq$ 0,
and, for convenience, define a function Q so that

$$Q(x, t) = S(x+a, t)$$

The initial condition on Q is then

$$Q(x, 0) = G_s\left[\frac{1}{a} - \frac{1}{2(x+a)} - \frac{1}{2(x+a)^2}\right] \quad x > 0 \tag{41}$$

The function Q(x, t) satisfies the Diffusion Equation in x and t and the boundary condition Q(0, t) = 0. As explained in ref. 6, a general solution for Q(x, t) may be expressed as

$$Q(x, t) = \frac{1}{2\sqrt{\pi k t}}\int_0^{+\infty} Q(x^1, 0)\left[e^{\frac{-(x-x^1)^2}{4kt}} - e^{\frac{-(x+x^1)^2}{4kt}}\right]dx^1 =$$

$$= \frac{G_s}{2\sqrt{\pi k t^1}}\int_0^{\infty}\left(\frac{1}{a} - \frac{1}{2(x^1+a)} - \frac{a}{2(x^1+a)^2}\right)\left(e^{\frac{-(x-x^1)^2}{4kt}} - e^{\frac{-(x+x^1)^2}{4kt}}\right)dx' \tag{42}$$

By substituting x for r - a in (42), one obtains the solutions for S(r, t):

$$S(r, t) = \frac{G_s}{2\sqrt{\pi k t}}\int_0^{\infty}\left[\frac{1}{a} - \frac{1}{2(x^1+a)} - \frac{G}{2(x^1+a)^2}\right]\left[e^{\frac{(r-a-x^1)^2}{4kt}} - e^{\frac{-(r-a+x^1)^2}{4kt}}\right]dx'$$

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